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On isolated components of ideals in multiplicative systems

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In [6] D. C. Murdoch has defined a right n-system associated with the McCoy's m-system, and extended, by using this concept, the results of Krull's isolated component ideals in commutative rings to the properties of the upper and lower right (or left) isolated component ideals in noncommutative rings. Thereafter W. E. Barnes [1] has given more different definitions of upper and lower right (or left) isolated component ideals of noncommutative rings, and obtained the analogous results of Murdoch by using the concept of the B-ν-system corresponding to the Murdoch's M-n-system. Moreover he has extended the Krull's "Hauptdarstellung" of ideals in commutative rings to the noncommutative case.

In the present paper we shall give decompositions of any ideal of an associative multiplicative system S into its upper and lower right (or left) principal components, and into its upper and lower right (or left) isolated component ideals by introducing the concept of the $\mathfrak{b}$-ν^*-system of S. This concept, which is fundamental in our study, is so well defined as to enable us to give our discussion lattice-theoretically, but it is considerably different from those of Murdoch and Barnes when S is a ring.

§**1.** Let K be a lattice-ordered semigroup (l-semigroup)[1] with the following conditions:

C_1) K is complete (upper and lower).

C_2) K has the greatest element e.

C_3) K has the zero element 0.

C_4) $ab \leq a$ and $ab \leq b$ for any two elements a and b of K.

We do not assume the greatest element e to be unity quantity (with respect to multiplication). But if e is unity quantity the condition C_4) holds for K.

Now we shall fix throughout this paper an accessible join-generator system[2] Σ of K.

Let a be an element of K. By the symbol $\Sigma[a]$ we shall mean the set of

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1) Cf. [2; p. 201].

2) A subset Σ of K is called here an accessible join-generator system of K, if it satisfies the conditions P_1) and P_2) in [5; p. 105]. Evidently K itself is one of its own accessible join-generator system.

the elements x of Σ such as $x\leq a$. The symbols ($\vee$), ($\wedge$) and ($-$) will denote respectively the set-theoretic union, intersection and difference. If $A_\wedge B$ is not void for two subsets A and B of K, we shall say frequently that A meets B or B meets A. By $\{a\,;\, a$ has property $P\}$ we mean the set of all elements a having property P. Evidently the set $X_{a,b}=\{c\,;\, ceb\leq a,\ c\in K\}$ is not void for any two elements a and b of K. The supremum of $X_{a,b}$ is called a right residual of a by b, which is somewhat different from the Birkhoff's residual[3]. Sup $[X_{a,b}]$ is denoted by $(a:b)_r$. Similarly for the left residual $(a:b)_l$ of a by b. K forms therefore a residuated lattice. If $(a:b)_r=a$, b is called relatively right prime to a. If every element of a subset B of K is relatively right prime to a, we say following Barnes[4] that B is entirely relatively right prime to a, or shortly B is ERP to a. By the symbol $M(a)$ we mean the set of the elements x of Σ such that $(a:x)_r=a$.

§ 2. Upper isolated components.

DEFINITION 1. Let a and b be two elements of K such as $a\leq b$. The (right) upper isolated b-component of a is the meet of the elements c of K which satisfy (1°) $c\geq a$ and (2°) $M(b)$ is ERP to c. In symbol: $U(a, b)$.

It is easily verified that if $b=p$ ($\neq e$) is a prime element containing a, the condition (2°) is equivalent to the condition[5]: $u\not\leq p$ ($u\in\Sigma$) implies $(c:u)_r=c$.

LEMMA 1. *If $a\leq a'\leq b$, then $U(a, b)\leq U(a', b)$.*

Proof. Let $S(a, b)$ be the set of the elements c of K which satisfy the conditions (1°) and (2°) in Definition 1. It is then easy to see that $S(a, b)\supseteq S(a', b)$. Hence we have $U(a, b)=\inf[S(a, b)]\leq\inf[S(a', b)]=U(a', b)$.

DEFINITION 2. Let b be an element which is not equal to e. If $M(b)$ is non-void, a subset N of Σ is called a (right) b-ν^*-system of K when

1*) $N\supseteq M(b)$,

2*) for every u of N and v of $M(b)$ there exists an element x of Σ such that $\Sigma[uxv]$ meets N.

If $M(b)$ is void, every subset of Σ is defined to be a b-ν^*-system of K. If $b=e$, every subset of Σ is also defined to be an e-ν^*-system of K.

LEMMA 2. *$N[b]=\Sigma-\Sigma[b]$ forms a b-ν^*-system for every element b of K.*

Proof. If $b=e$, the lemma is evident by the definition. We now suppose that b is not equal to e. Take an arbitrary element v of $M(b)$. Then $v\not\leq b$. For, if $v\leq b$, we have $b=(b:v)_r\geq(b:b)_r=e$, $b=e$, a contradiction. Hence $v\notin\Sigma[b]$, that is, $M(b)$ does not meet $\Sigma[b]$. $M(b)$ is therefore contained in $N[b]$. Take arbitrary

3) Cf. [2; p. 201] and [5; p. 105].

4) Cf. [1; p. 11].

5) Cf. [5; p. 113].

elements y and z of $N[b]$ and $M(b)$ respectively. Then there exists an element x of Σ such that $\Sigma[yxz]$ meets $N[b]$. For, if not so, i.e., if $\Sigma[yxz]$ does not meet $N[b]$ for every element x of Σ, then evidently $\Sigma[yxz]$ is contained in $\Sigma[b]$ for every x of Σ. This implies that $yez=y(\sup[\Sigma])z=\sup_x[yxz]\leq b$. Hence we have $y\leq(b:z)_r=b$, $y\in\Sigma[b]$. Therefore $N[b]\wedge\Sigma[b]$ is not void. This is a contradiction.

LEMMA 3. *Let a be an element of K such that $\Sigma[a]$ does not meet a b-ν*-system N. Then there exists an element q of K such that* (1) $q\geq a$, (2) *$M(b)$ is ERP to q and* (3) *$\Sigma[q]$ does not meet N.*

Proof. Take an ascending chain $a\leq a_1\leq a_2\leq\cdots$ of K such that $\Sigma[a_i]$ does not meet N for $i=1, 2, \cdots$. Then $\Sigma[a^*]$ does not meet N, where $a^*=\sup_i[a_i]$. For, if not so, then there exists an element u of N such as $u\leq a^*$. Since it is easily verified that $a^*=\sup[\bigvee_{i=1}^{\infty}\Sigma[a_i]]$, there exists a finite number of elements $u_1, \cdots, u_\rho$ satisfying $u\leq u_1\cup\cdots\cup u_\rho$, $u_j\in\bigvee_{i=1}^{\infty}\Sigma[a_i]$ $(j=1, \cdots, \rho)$. Hence $u_1\cup\cdots\cup u_\rho\leq a_\sigma$ for a sufficiently large whole number σ. Hence $u\leq a_\sigma$, $u\in\Sigma[a_\sigma]$. This is a contradiction. Zorn's lemma assures therefore the existence of maximal element q with the conditions (1) and (3).

Next we prove that q satisfies the condition (2). Take an element y of Σ such as $y\not\leq q$. Then $\Sigma[q\cup y]$ meets N. Take now elements u of $\Sigma[q\cup y]\wedge N$ and v of $M(b)$. Then $uxv\leq(q\cup y)xv=qxv\cup yxv\leq q\cup yev$, If $yev\leq q$, then $uxv\leq q$. Hence $\Sigma[uxv]\subseteq\Sigma[q]$ for every x of Σ. On the other hand there exists an element x_0 of Σ such that $\Sigma[ux_0v]$ meets N, a contradiction. Hence we get $yev\not\leq q$. In other words $yev\leq q$ implies $y\leq q$. We get therefore $q\geq(q:v)_r\geq q$, $q=(q:v)_r$ for every element v of $M(b)$.

THEOREM 1. *Let a and b be two elements of K such as $a\leq b$. Then $N[U(a, b)]=\Sigma-\Sigma[U(a, b)]$ is a unique maximal b-ν*-system which does not meet $\Sigma[a]$.*

Proof. Suppose that $N\equiv N[U(a, b)]$ is not void. Take an element u of N. Then $u\not\leq U(a, b)$. Hence there exists an element c of K such that (0) $c\not\geq u$, (1) $c\geq a$ and (2) $M(b)$ is ERP to c. Let v be an element of $M(b)$. Then it is easily verified that there exists an element x_0 of Σ such as $ux_0v\not\leq c$. We now suppose that $\Sigma[ux_0v]$ does not meet N. Then $\Sigma[ux_0v]\subseteq\Sigma[U(a, b)]\subseteq\Sigma[c]$. This implies $ux_0v=\sup[\Sigma[ux_0v]]\leq\sup[\Sigma[c]]=c$, a contradiction. Hence $\Sigma[ux_0v]$ meets N. This proves the condition 2*) of Definition 2. Next we prove that $M(b)$ is contained in N. If there exists an element v of $M(b)$ which is not contained in N, then $v\leq c$ for every element c with the properties (1°) and (2°) in Definition 1. Hence $c=(c:v)_r\geq(c:c)_r=e$, $c=e$ for every c, hence $U(a, b)=e$, and hence N is void, which is a contradiction. N is clearly disjoint from $\Sigma[a]$. Next let N' be an arbitrary b-ν*-system disjoint from $\Sigma[a]$. Then there exists, by Lemma 3, an element q of K such that (1) $q\geq a$, (2) $M(b)$ is ERP to q and (3) $\Sigma[q]$ does not meet N'. Since $U(a, b)\leq q$, we obtain $N\supseteqq\Sigma-\Sigma[q]\supseteqq N'$. This completes the proof.

THEOREM 2. *$U(a, b)$ is equal to the join of all the elements w of Σ such that every b-ν^*-system containing w contains an element of $\Sigma[a]$.*

Proof. Let W be the set of all elements w mentioned in the theorem. Since $w\leq U(a, b)$ for every w of W, we have $\sup[W]\leq U(a, b)$. Suppose that $\sup[W]\neq U(a, b)$. Then it is easily verified that there exists an element x of Σ such as $x\leq U(a, b)$ and $x\not\leq \sup[W]$. Take now a b-ν^*-system N' which is disjoint from $\Sigma[a]$ and contains x. Then N' is contained in $N[U(a, b)]$. Hence $x\not\leq U(a, b)$, which is a contradiction.

LEMMA 4. *If a is a primal element with prime adjoint[6] p, then $U(a, p)=a$.*

Proof. This is immediate by Lemma 25 in [5].

THEOREM 3. *Let $a=\bigcap_{\lambda\in\Lambda}a_\lambda$ be a decomposition of a into strongly meet irreducible elements a_λ, and let $p_\lambda=adj(a_\lambda)$. Then $a=\bigcap_{\lambda\in\Lambda}U(a, p_\lambda)$.*

Proof. By Lemmas 20 in [5] and 4, we have $a_\lambda=U(a_\lambda, p_\lambda)$. Since $a\leq a_\lambda$ and $a_\lambda\leq p_\lambda$, we obtain $a=\bigcap_\lambda a_\lambda=\bigcap_\lambda U(a_\lambda, p_\lambda)\geq\bigcap_\lambda U(a, p_\lambda)\geq a$, $a=\bigcap_\lambda U(a, p_\lambda)$.

THEOREM 4. *Suppose that K is modular as a lattice. If the ascending chain condition holds for the elements of K, then every element a of K is decomposed as*

$$a = U(a, p_1)\cap\cdots\cap U(a, p_n)\,,$$

where $p_1, \cdots, p_n$ are the maximal primes[7] of a.

Proof. Let $a=q_1\cap\cdots\cap q_n$ be a decomposition of a into primal elements q_i with $p_i=\text{adj}\,(q_i)$. Then $a\leq U(a, p_i)\leq U(q_i, p_i)=q_i$. Hence we obtain $a\leq\bigcap_{i=1}^n U(a, p_i)\leq\bigcap_{i=1}^n q_i=a$, $a=\bigcap_{i=1}^n U(a, p_i)$.

LEMMA 5. *Suppose that K is modular as a lattice. If the adjoint p_λ of a strongly meet-irreducible element a_λ containing a is relatively (right) prime to a, then $U(a, p_\lambda)$ is redundant in $a=\bigcap_{\lambda\in\Lambda}U(a, p_\lambda)$.*

Proof. Let $a=\bigcap_\lambda a_\lambda$ be a decomposition of a into strongly meet-irreducible primal elements a_λ, and let $p_\lambda=\text{adj}\,(a_\lambda)$. Then we have $a=\bigcap_\lambda U(a, p_\lambda)$ by Theorem 3. If a_λ is redundant, then so is $U(a, p_\lambda)$. Now we suppose that $U(a, p_\kappa)$ is irredundant. Then a_κ is also irredundant. Hence $a_\kappa\not\geq\bigcap_{\lambda\neq\kappa}a_\lambda\equiv a_\kappa^*$ and $a_\kappa\cup a_\kappa^*>a_\kappa$. On the other hand, since p_κ is an NRP-element[8] of a_κ, we can take an element x of Σ such that $x\not\leq a_\kappa$ and $xe p_\kappa\leq a_\kappa$. Hence $a_\kappa\neq a_\kappa\cup x\equiv a_\kappa'$. Since

6) Cf. [5; p. 107].
7) Cf. [5; p. 106].
8) Cf. Lemma 18 in [5].

a_κ is meet-irreducible, we have $a_\kappa \neq a'_\kappa \cap (a_\kappa \cup a^*_\kappa) = a_\kappa \cup (a'_\kappa \cap a^*_\kappa)$. Hence we can take an element y of Σ such that $y \not\leq a_\kappa$, $y \leq a'_\kappa$ and $y \leq a^*_\kappa$. $y \leq a'_\kappa$ implies the existence of a finite number of elements $z^{(1)}_\kappa, \cdots, z^{(\rho_\kappa)}_\kappa$ of Σ such as $z^{(i)}_\kappa \leq a_\kappa$ and $y \leq z^{(1)}_\kappa \cup \cdots \cup z^{(\rho_\kappa)}_\kappa \cup x$. Hence we have $yep_\kappa \leq \bigcup_{i=1}^{\rho_\kappa} (z^{(i)}_\kappa ep_\kappa) \cup xyep_\kappa \leq a_\kappa \cup xep_\kappa = a_\kappa$. On the other hand, $y \leq a^*_\kappa$ implies $yep_\kappa \leq a_\kappa$ for $\kappa \neq \lambda$. Thus we obtain $yep_\kappa \leq a$; that is, p_κ is an NRP-element of a. Hence if p_κ is relatively prime to a, $U(a, p_\kappa)$ is redundant. This completes the proof.

§ 3. Lower isolated components, principal components.

DEFINITION 3. Let a and b be two elements of K such that $a \leq b$. If $M(b)$ is non-void, the supremum of the set $\{z\,;\, z \leq (a:v)_r$ for some $v \in M(b),\ z \in \Sigma\}$ is called a (right) lower isolated b-component of a. In symbol: $L(a, b)$. If $M(b)$ is void, $L(a, b)$ is defined to be a. $L(a, e)$ is also defined to be a.

LEMMA 6. *If $a \leq b$, then $a \leq L(a, b) \leq U(a, b)$.*

Proof. If $M(b)$ is void, then $a = L(a, b) = U(a, b)$. Evidently $a = L(a, e) = U(a, e)$. We now suppose that $b \neq e$ and $M(b)$ is not void. Take an arbitrary element x of Σ such as $x \leq L(a, b)$. Then we can find a finite number of elements $x_1, \cdots, x_\rho$ of Σ such as $x \leq x_1 \cup \cdots \cup x_\rho$, $x_i \leq (a : v_i)_r$, where $v_i \in M(b)$. Since $x_i e v_i \leq a$, i.e., $\Sigma[x_i e v_i] \subseteq \Sigma[a]$ for $i = 1, \cdots, \rho$, every b-ν^*-system containing x_i meets $\Sigma[a]$. Then we have by Theorem 4 that $x_i \leq U(a, b)$ for every i. Hence we get $x \leq U(a, b)$, i.e., $L(a, b) \leq U(a, b)$. $a \leq L(a, b)$ is immediate by the definition of $L(a, b)$.

THEOREM 5. *Let $a = \bigcap_{\lambda \in \Lambda} a_\lambda$ be a decomposition of an element a of K into strongly meet-irreducible primal elements a_λ, and let $p_\lambda = adj\,(a_\lambda)$. Then $a = \bigcap_{\lambda \in \Lambda} L(a, p_\lambda)$.*

Proof. This is immediate by Theorem 3 and Lemma 6.

THEOREM 6. *Suppose that K is modular as a lattice. If the ascending chain condition holds for the elements of K, every element a of K is decomposed as*

$$a = L(a, p_1) \cap \cdots \cap L(a, p_n)\,,$$

where $p_1, \cdots, p_n$ are the maximal primes of a.

Proof. This is immediate by Theorem 4 and Lemma 6.

THEOREM 7.

$$\begin{aligned} &(1) \quad U(U(a, b), b) = U(a, b)\,,\\ &(2) \quad L(U(a, b), b) = U(a, b)\,,\\ &(3) \quad U(L(a, b), b) = U(a, b)\,. \end{aligned}$$

Proof. (1) : $N[U(a, b)]$ is a b-ν^*-system which is disjoint from $\Sigma[U(a, b)]$. Hence it is maximal in such systems, and hence by Theorem 2 $\sup[\Sigma - N[U(a, b)]] = \sup[\Sigma[U(a, b)]]$ is the upper isolated b-component of $U(a, b)$. By (1) and

Lemma 6 we can prove (2); and by (1) and Theorem 2 we can prove (3), which are similar as in the case of rings[9].

For all ordinal numbers α we define $L^{\alpha}(a, b)$ by induction as follows: $L^{1}(a, b)=L(a, b)$. If α is not a limit ordinal, $L^{\alpha}(a, b)=L(L^{\alpha-1}(a, b), b)$, while if α is a limit ordinal, $L^{\alpha}(a, b)$ is the supremum of all $L^{\sigma}(a, b)$ for each $\sigma<\alpha$. Then the following properties are proved which are quite analogous to the case of rings[10]. (1) For all ordinal numbers α, $U(a, b)\geq L^{\alpha}(a, b)$. (2) For an ordinal number α, $L^{\alpha}(a, b)=L^{\alpha+1}(a, b)$ if and only if $L^{\alpha}(a, b)=U(a, b)$. (3) There exists an ordinal number α, finite or infinite, such that $L^{\alpha}(a, b)=U(a, b)$. (4) If the ascending chain condition holds for the closed interval $[e, a]$, then $L^{n}(a, b)=U(a, b)$ for some finite number n.

DEFINITION 4. Let a and q be two elements of K such as $a\leq q$. If q is a W-maximal element[11] of a, $L(a, q)$ is called a (right) lower principal component of a, and $U(a, q)$ is called a (right) upper principal component of a[12].

THEOREM 8. *Every element of K is represented as the meet of its lower principal components.*

Proof. Let $\{q_{\lambda}; \lambda\in\Lambda\}$ be the set of all the W-maximal NRP-elements of a. If either $e=q_{\lambda}$ for some λ or e is weakly NRP to q_{λ} for some λ, the results follows trivially. Now we suppose that $e\neq q_{\lambda}$ for all λ of Λ. Take an element x of Σ such as $x\leq\bigcap_{\lambda\in\Lambda}L(a, q_{\lambda})$. Then we can find a finite number of elements $z_{1}, \cdots$, z_{ρ} such that $x\leq z_{1}\cup\cdots\cup z_{\rho}$, $z_{i}\in\{z; z\leq(a:v)_{r}$ for some $v\in M(q_{\lambda})\}$ $(i=1, \cdots, \rho)$. Hence $z_{i}\leq(a:v_{i})_{r}$, i.e., $z_{i}ev_{i}\leq a$ $(i=1, \cdots, \rho)$. Now since $xe(v_{1}ev_{2}e\cdots ev_{\rho})\leq\bigcup_{i=1}^{\rho}z_{i}e\times(v_{1}ev_{2}e\cdots ev_{\rho})\leq z_{1}ev_{1}\cup z_{2}ev_{2}\cup\cdots\cup z_{\rho}ev_{\rho}\leq a$, the element $v_{1}ev_{2}e\cdots ev_{\rho}$ is contained in the set $F\equiv\{f; xef\leq a, f\in K\}$. Hence $v_{1}ev_{2}e\cdots ev_{\rho}\leq\sup[F]$. If we suppose that $\sup[F]\leq q_{\lambda}$ for some λ of Λ, then $(v_{1}ev_{2}e\cdots ev_{\rho-1})ev_{\rho}\leq q_{\lambda}$. Hence $v_{1}ev_{2}e\cdots ev_{\rho-1}\leq(q_{\lambda}:v_{\rho})_{r}=q_{\lambda}$. Continuing in this way, we obtain $v_{1}\leq q_{\lambda}$. This is a contradiction. Hence $\sup[F]\not\leq q_{\lambda}$ for every λ of Λ, hence $\sup[F]\not\leq a$, and hence $(a:\sup[F])_{r}=a$. Since $xe\cdot\sup[F]\leq a$, we obtain $x\leq a$, i.e., $\bigcap_{\lambda\in\Lambda}L(a, q_{\lambda})\leq a$. The converse inclusion is evident. This completes the proof.

THEOREM 9. *Suppose that K is modular. If an element a of K is represented as a meet of a finite or infinite number of upper isolated p_{λ}-components of a, where $p_{\lambda}=adj(a_{\lambda})$ and a_{λ} are strongly meet-irreducible primal element of a, then a is equal to the meet of its upper principal components.*

Proof. By the proof of Lemma 5 each p_{λ} is an NRP-element of a. Take a

9) Cf. [6; p. 47].
10) Cf. [1; p. 13] or [6; p. 47].
11) Cf. [5; p. 106].
12) Cf. [3; p. 16].

W-maximal NRP-element p_λ^* of a such that $p_\lambda \leq p_\lambda^*$ for each λ. If $p_\lambda^* = e$, then $U(a, p_\lambda^*) = a \leq U(a, p_\lambda)$. If $p_\lambda^* \neq e$, then $N[p_\lambda]$ contains the set $\boldsymbol{P}_\lambda$ consisting of all elements of Σ which are relatively prime to p_λ^*. If $N[p_\lambda]$ is NRP to an element b of K such as $b \geq a$, then $\boldsymbol{P}_\lambda$ is of course ERP to b. Hence $U(a, p_\lambda) \geq U(a, p_\lambda^*) \geq a$. We obtain therefore $a = \bigcap_\lambda U(a, p_\lambda^*)$, completing the proof.

§4. Isolated component ideals in associative m-systems

Let $\mathfrak{A}$ be an algebraic system with a void or non-void (finite or infinite) set Γ of the finitary operations $\varphi, \psi, \cdots$. In this section a closed subsystem under every operation in Γ is called a subalgebra of $\mathfrak{A}$. The subalgebra generated by the subset X of $\mathfrak{A}$ is denoted by $[X]^\Gamma$. Let $a_1, \cdots, a_n$ be a finite number of elements of $\mathfrak{A}$. We denote by $f_{\varphi_1, \cdots, \varphi_m}(a_1, \cdots, a_n)$ or losely by $f(a_1, \cdots, a_n)$ the element of $[a_1, \cdots, a_n]^\Gamma$. The subalgebra generated by $\mathfrak{B}$ and $\mathfrak{C}$ is denoted by $\mathfrak{B} \cup \mathfrak{C}$, where $\mathfrak{B}$, $\mathfrak{C}$ are two subalgebras of $\mathfrak{A}$.

Let $\mathfrak{B}$ be a subalgebra, and let $\{b_1, b_2, \cdots\}$ be any fixed generator-system of $\mathfrak{B}$. If $b \in \mathfrak{B}$, then $b = f(b_{i_1}, \cdots, b_{i_n})$. Evidently $f(b_{i_1}, \cdots, b_{i_n})$ is contained in $[b_{i_1}, \cdots, b_{i_n}]^\Gamma = [[b_{i_1}]^\Gamma \vee \cdots \vee [b_{i_n}]^\Gamma]^\Gamma$. Hence $[b]^\Gamma$ is contained in $[b_{i_1}]^\Gamma \cup \cdots \cup [b_{i_n}]^\Gamma$. This proves that the set of all subalgebras, each of which is generated by a single element, forms an accessible join-generator system of the lattice of all subalgebras of $\mathfrak{A}$.

An associative multiplicative semigroup R is called a ring system if (1) R is an algebra with Γ which does not contain the multiplication and (2) the multiplication is distributive with respect to each operation in Γ. Usual semigroups, rings and distributive lattices are included as very special cases. A subset $\mathfrak{a}$ of R is called a (two-sided) Γ-ideal or shortly an ideal of R if (1) $\mathfrak{a}$ is a subalgebra with respect to Γ and (2) $\mathfrak{a}$ contains Ra[13] and aR for every element a of $\mathfrak{a}$. The ideal generated by a subset X of R is equal to the subalgebra $[RXR \vee RX \vee XR \vee [X]^\Gamma]^\Gamma = [RXR \vee RX \vee XR \vee X]^\Gamma$, which is denoted by $(X)^\Gamma$. For a single element u, $(u)^\Gamma$ is called a principal ideal of R.

The set of all ideals of a ring system R forms an upper and lower complete l-semigroup under the set inclusion relation and the multiplication $\mathfrak{a} \cdot \mathfrak{b} = (\mathfrak{a}\mathfrak{b})^\Gamma$, where $\mathfrak{a}$ and $\mathfrak{b}$ are any two ideals of R. Hence it is also residuated with respect to the residuals: $(\mathfrak{a} : \mathfrak{b})_r = (X_{\mathfrak{a},\mathfrak{b}})^\Gamma$ and $(\mathfrak{a} : \mathfrak{b})_l = (Y_{\mathfrak{a},\mathfrak{b}})^\Gamma$, where $X_{\mathfrak{a},\mathfrak{b}} = \{x ; xR\mathfrak{b} \subseteq \mathfrak{a}\}$ and $Y_{\mathfrak{a},\mathfrak{b}} = \{y ; \mathfrak{b}Ry \subseteq \mathfrak{a}\}$.

LEMMA 7. *The set of all principal ideals of any ring system R forms an accessible join-generator system of the lattice of all ideals of R.*

Proof. Let $\mathfrak{a}$ be any ideal of R, and $\{a_1, a_2, \cdots\}$ any fixed generator-system of $\mathfrak{a}$. Then an element x of $\mathfrak{a}$ is represented as follows: $x = f(a_1, \cdots, a_n)$. Hence $x \in [a_1]^\Gamma \cup \cdots \cup [a_n]^\Gamma = [\bigvee_{i=1}^n a_i]^\Gamma$. We have therefore

13) $XY = \{xy ; x \in X, y \in Y\}$; in particular $Xa = \{xa ; x \in X\}$.

$$\begin{aligned}(x)^{\Gamma} &= [RxR \vee Rx \vee xR \vee x]^{\Gamma}\\ &\subseteq [[\bigvee_{i=1}^{n} Ra_iR]^{\Gamma} \vee [\bigvee_{i=1}^{n} Ra_i]^{\Gamma} \vee [\bigvee_{i=1}^{n} a_iR]^{\Gamma} \vee [\bigvee_{i=1}^{n} a_i]^{\Gamma}]^{\Gamma}\\ &= [\bigvee_{i=1}^{n} Ra_iR \vee \bigvee_{i=1}^{n} Ra_i \vee \bigvee_{i=1}^{n} a_iR \vee \bigvee_{i=1}^{n} a_i]^{\Gamma}\\ &= [\bigvee_{i=1}^{n} (Ra_iR \vee Ra_i \vee a_iR \vee a_i)]^{\Gamma}\\ &= [\bigvee_{i=1}^{n} [Ra_iR \vee Ra_i \vee a_iR \vee a_i]^{\Gamma}]^{\Gamma}\\ &= [\bigvee_{i=1}^{n} (a_i)^{\Gamma}]^{\Gamma} = (a_1)^{\Gamma} \cup \cdots \cup (a_n)^{\Gamma}.\end{aligned}$$

This completes the proof.

Now let A be any s-ideal of R, and let $\bar{A}$ be the set of all elements $f(a_1, \cdots, a_n)$, $a_i \in A$. Then the mapping $A \to \bar{A}$ satisfies the conditions 1), $\cdots$, 6) in [5; § 12].

In the following S will denote an associative multiplicative system (l-semigroup) with zero. Let $A \to \bar{A}$ be a mapping from S into itself with the conditions 1), $\cdots$, 6) in [5; § 12]. A subset $\mathfrak{a}$ of S is called an ideal of S, if $S\mathfrak{a} \subseteq \mathfrak{a}$, $\mathfrak{a}S \subseteq \mathfrak{a}$ and $\bar{\mathfrak{a}} = \mathfrak{a}$.

We now assume throughout the rest of this section that the set of all principal ideals of S forms an accessible join-generator system of the lattice of all ideals of S.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be any two ideals of S, and let $\mathfrak{c}$ be the ideal generated by the set-union of all ideals $\mathfrak{x}$ such that $\mathfrak{x}S\mathfrak{b}$ is contained in $\mathfrak{a}$. Then by 4) $\mathfrak{c}$ is the greatest ideal satisfying $\mathfrak{c}S\mathfrak{b} \subseteq \mathfrak{a}$, which is defined to be the right residual of $\mathfrak{a}$ by $\mathfrak{b}$ and denoted by $(\mathfrak{a} : \mathfrak{b})_r$. Similarly for a left residual of $\mathfrak{a}$ by $\mathfrak{b}$. If $(\mathfrak{a} : \mathfrak{b})_r = \mathfrak{a}$, $\mathfrak{b}$ is called relatively (right) prime to $\mathfrak{a}$. A family $\boldsymbol{B}$ of ideals of S is called entirely relatively (right) prime to $\mathfrak{a}$, or shortly ERP to $\mathfrak{a}$, if every ideal in $\boldsymbol{B}$ is relatively prime to $\mathfrak{a}$. By the symbol $\boldsymbol{M}(\mathfrak{a})$ we mean the family of principal ideals which are relatively prime to $\mathfrak{a}$.

Let $\mathfrak{a}$ be an ideal of S, and $\mathfrak{b}$ an ideal containing $\mathfrak{a}$. The (right) upper isolated $\mathfrak{b}$-component $U(\mathfrak{a}, \mathfrak{b})$ of $\mathfrak{a}$ is the intersection of ideals $\mathfrak{c}$ such that $\mathfrak{c}$ contains $\mathfrak{a}$ and $\boldsymbol{M}(\mathfrak{b})$ is ERP to $\mathfrak{c}$.

Let A be any subset of S. Then the set-union A^p of all principal ideals (x), $x \in A$, is called a p-closure of A. A subset A of S is called p-closed if $A^p = A$. It is then easily verified that the set-union of principal ideals is p-closed and conversely. Let $\mathfrak{b}$ be an ideal which is different from S. If $\boldsymbol{M}(\mathfrak{b})$ is non-void, a p-closed subset N of S is called a (right) $\mathfrak{b}$-ν^*-system of S, when 1) every ideal in $\boldsymbol{M}(\mathfrak{b})$ is contained in N and 2)[14] for every (u) in N and (v) in $\boldsymbol{M}(\mathfrak{b})$ there exists (x) such that $(u)(x)(v) \wedge N$ is not void. If $\boldsymbol{M}(\mathfrak{b})$ is void, every p-closed

14) Evidently the condition 2) is weaker than the condition 2*): For every element n of N and m of $\{z ; z \in (v)$ for some $(v) \in \boldsymbol{M}(\mathfrak{b})\}$, there exist three elements s, t and u of S such as $sntmu \in N$. In order to extend some results of Murdoch [6] to an l-semigroup, it is convenient to define the right M'-n-system of a ring $\mathfrak{o}$ as a subset (not necessarily p-closed) of $\mathfrak{o}$ which contains an m-system M' in the sense of M. Nagata [7] and satisfies the condition obtained by the substitution of M' in place of $\{z ; \cdots\}$ in the condition 2*).

subset of S is a $\mathfrak{b}$-ν^*-system of S. If $\mathfrak{b}=S$, every p-closed subset of S is also a $\mathfrak{b}$-ν^*-system of S. The complement $C(\mathfrak{b})$ of an ideal $\mathfrak{b}$ is not necessarily a $\mathfrak{b}$-ν^*-system. The p-closure of $C(\mathfrak{b})$ forms a $\mathfrak{b}$-ν^*-system of S, since the p-closure of $C(\mathfrak{b})$ is equal to the set-union of the principal ideals, each of which is not contained in $\mathfrak{b}$ (Lemma 2). Let $\mathfrak{a}$ be an ideal which does not meet a $\mathfrak{b}$-ν^*-system N. Then by Lemma 3 we can find an ideal $\mathfrak{q}$ such that 1) $\mathfrak{q}\supseteq\mathfrak{a}$, 2) $\boldsymbol{M}(\mathfrak{b})$ is ERP to $\mathfrak{q}$ and 3) $\mathfrak{q}$ does not meet N. By Theorem 1 the p-closure of the complement of the upper isolated $\mathfrak{b}$-component of an ideal $\mathfrak{a}$ forms a unique maximal $\mathfrak{b}$-ν^*-system disjoint from $\mathfrak{a}$.

Now we remark that $U(\mathfrak{a}, \mathfrak{b})$ is equal to the set-union of all the elements x of S such that every $\mathfrak{b}$-ν^*-system containing x contains an element of $\mathfrak{a}$. Because, $U(\mathfrak{a}, \mathfrak{b})$ is equal to the set-union of all principal ideals (x) such that every $\mathfrak{b}$-ν^*-system containing (x) contains a principal ideal contained in $\mathfrak{a}$. Let U' be the set-union of all the elements x such that every $\mathfrak{b}$-ν^*-system N containing x contains an element a of $\mathfrak{a}$. Then evidently (x) is contained in N, and (a) is contained in N and $\mathfrak{a}$. Hence U' is contained in $U(\mathfrak{a}, \mathfrak{b})$. The converse inclusion is easy to see.

The primal ideal, its adjoint ideal, the (right) lower isolated component ideal, etc. are defined in the obvious way. Hence Theorems 3, 5 and 7 are applicable to the ideals of S, which are different from the Barnes' results when S is a ring.

For a W-maximal[15] ideal $\mathfrak{q}$ of S, the (right) lower principal component ideal $L(\mathfrak{a}, \mathfrak{q})$ and the upper principal component ideal $U(\mathfrak{a}, \mathfrak{q})$ of an ideal $\mathfrak{a}$ of S are also difined (Definition 4). Then Theorem 8 is applicable to the ideals of S. If the lattice of all the ideals of S is modular, Theorem 9 is valid for the ideals of S. Moreover if the ascending chain condition holds for the ideals of S, Theorems 4 and 6 are applicable to the ideals of S.

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15) Cf. [5; p. 106].